Finite element methods respecting the discrete maximum principle for convection-diffusion equations

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Convection-diffusion-reaction equations model the conservation of scalar quantities. From the analytic point of view, solution of these equations satisfy under certain conditions maximum principles, which represent physical bounds of the solution. That the same bounds are respected by numerical approximations of the solution is often of utmost importance in practice. The mathematical formulation of this property, which contributes to the physical consistency of a numerical method, is called Discrete Maximum Principle (DMP). In many applications, convection dominates diffusion by several orders of magnitude. It is well known that standard discretizations generally do not satisfy the DMP in this convection-dominated regime, where solutions typically exhibit layers, which are very thin regions with a steep gradient. In this case, one faces a multiscale problem with the layers being subgrid scales that cannot be resolved on affordable grids. It turns out to be a challenging problem to construct discretizations that, on the one hand, respect the DMP and, on the other hand, compute accurate solutions.

This talk presents a survey on finite element methods, with a main focus on the convection-dominated regime of the steady-state problem, that satisfy a local or a global DMP. It reveals that for the steady-state problem there are only a few discretizations, all of them nonlinear, that at the same time satisfy the DMPs and compute reasonably accurate solutions, e.g., algebraically stabilized schemes. Moreover, most of these discretizations have been developed in recent years, showing the enormous progress that has been achieved lately. A brief discussion of the evolutionary problem will be provided as well.

This talk is joint work with Gabriel R. Barrenechea (Glasgow) and Petr Knobloch (Prague) and it is based on [1].

References

[1] Gabriel R. Barrenechea, Volker John, and Petr Knobloch. Finite element methods respecting the discrete maximum principle for convection-diffusion equations. *SIAM Rev.* 66, 3–88, 2024.