The Stokes system with traction boundary conditions in 3D exterior domains

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Let $\Omega \subset \mathbb{R}^3$ be open, bounded, with C^2-boundary denoted by $\partial\Omega$. Put $\overline{\Omega}^c := \mathbb{R}^3 \backslash \overline{\Omega}$ (exterior domain). Consider the Stokes system with traction boundary conditions,

$$-\Delta u + \nabla \pi = f,\ \operatorname{div} u = 0 \ \text{ in } \overline{\Omega}^c, \quad \sum_{k=1}^{3} (\partial_j u_k + \partial_k u_j - \delta_{jk}\, \pi)\, n^{(\Omega)} = b_j \ \ (1 \le j \le 3). \quad (1)$$

Here $n^{(\Omega)}$ denotes the outward unit normal to Ω. Following up a suggestion by T. Hishida (University of Nagoya), we could show that in general, problem (1) admits two solutions, which differ insofar as the pressure part of one of them is L^γ-integrable near infinity for some $\gamma \in (1, \infty)$ if f has sufficient decay, whereas the velocity part of the other one satisfies a zero flux condition on $\partial\Omega$.

More precisely, let $q \in (1, \infty)$, $r \in (1, 3)$, $s \in (1,\, 3/2)$, $b \in W^{1-1/q,\, q}(\partial\Omega)^3$ and $f \in L^q(\overline{\Omega}^c)^3 \cap L^r(\overline{\Omega}^c)^3 \cap L^s(\overline{\Omega}^c)^3$. Then for $\nu \in \{1,\, 2\}$, there is a solution $(u^{(\nu)},\, \pi^{(\nu)}) \in W^{2,q}_{loc}(\Omega^c)^3 \cap W^{1,q}_{loc}(\Omega^c)$ of (1) such that

$$\partial_j \partial_k u^{(\nu)},\ \nabla \pi^{(\nu)} \in L^q(\overline{\Omega}^c)^3,\ \partial_j u^{(\nu)} \in L^{(1/r-1/3)^{-1}}(\overline{\Omega}^c)^3,\ u^{(\nu)} \in L^{(1/s-2/3)^{-1}}(\overline{\Omega}^c)^3$$

$(1 \le j, k \le 3)$, and in the case $\nu = 1$ in addition $\pi^{(\nu)}|B^c_R \in L^\gamma(B^c_R)$ for some $\gamma \in (3/2,\, \infty)$ and some $R \in (0, \infty)$ with $\overline{\Omega} \subset B_R$. In the case $\nu = 2$ the condition on $\pi^{(\nu)}|B^c_R$ is replaced by $\int_{\partial\Omega} u^{(\nu)} \cdot n^{(\Omega)}\, do_x = 0$. In general the difference $u^{(1)} - u^{(2)}$ is nonconstant.

Moreover the space of all functions $(u, \pi) \in W^{2,p}_{loc}(\Omega^c)^3 \times W^{1,p}_{loc}(\Omega^c)$ such that $u|B^c_R \in L^t(B^c_R)^3$ and $\nabla\pi|B^c_R \in L^\sigma(B^c_R)^3$ for some $p,\, t,\, \sigma \in (1, \infty)$ and some R as above, and in addition $\int_{\partial\Omega} u \cdot n^{(\Omega)}\, do_x = 0$, is a uniqueness class of solutions to (1). The same is true if the condition $\int_{\partial\Omega} u \cdot n^{(\Omega)}\, do_x = 0$ is replaced by $\pi|B^c_R \in L^\gamma(B^c_R)$ for some $\gamma \in (1, \infty)$ and some R as above.