On the Helmholtz decomposition in general domains

Werner Varnhorn
Institute of Mathematics, Kassel University, Germany
varnhorn@mathematik.uni-kassel.de

Joint work with Christian Simader and Hermann Sohr

In the theory of the incompressible Navier-Stokes equatuons the Helmholtz decomposition (HD) in $L^q(\Omega)$ plays a fundamental role. It holds if each $f \in L^q(\Omega)$ has the unique decomposition $f = f_0 + \nabla p$ with $f_0 \in L^q_\sigma(\Omega)$, $\nabla p \in G^q(\Omega)$, satisfying the estimate $\|f_0\|_q + \|\nabla p\|_q \leq C\,\|f\|_q$, $C = C(\Omega, q) > 0$.
Here $L^q_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_q}$ denotes the closure of smooth divergence free vector functions with compact support in Ω, and $G^q(\Omega) = \{\nabla p \in L^q(\Omega);\ p \in L^q_{loc}(\Omega)\} \subseteq L^q(\Omega)$ the gradient subspace of $L^q(\Omega)$.

Our investigation of (HD) is based on a gradient estimate (GE) in the form $\|\nabla p\|_q \leq C\ \sup |\langle \nabla p, \nabla v\rangle_\Omega|/\|\nabla v\|_{q'}$, $\nabla v \in G^{q'}(\Omega)$, $q' = \frac{q}{q-1}$, with some constant $C = C(\Omega, q) > 0$. This estimate was introduced in [1] for smooth bounded and exterior domains. We show for general domains $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, $1 < q < \infty$ that the validity of this gradient estimate (GE) in $G^q(\Omega)$ and in $G^{q'}(\Omega)$ is necessary and sufficient for the validity of the Helmholtz decomposition in $L^q(\Omega)$ and in $L^{q'}(\Omega)$.
A second approach concerns the estimate (DE) for divergence free functions $f_0 \in L^q_\sigma(\Omega)$ in the form $\|f_0\|_q \leq C \sup |\langle f_0, w\rangle_\Omega|/\|w\|_{q'}$, $w \in L^{q'}_\sigma(\Omega)$. We show for general domains that the validity of this estimate in $L^q_\sigma(\Omega)$ and in $L^{q'}_\sigma(\Omega)$ is also necessary and sufficient for the validity of the Helmholtz decomposition in $L^q(\Omega)$ and in $L^{q'}(\Omega)$.
Finally, we study the optimal constants in the estimates (GE) and (DE) and prove that these constants coincide.

References

[1] *C. G. Simader, H. Sohr*: A new approach to the Helmholtz decomposition and the Neumann problem in L^q-spaces for bounded and exterior domains. Series on Advances in Math. for Appl. Sci. 11, 1-35, World Scientific, Singapore, 1992.

[2] *C. G. Simader, H. Sohr, W. Varnhorn*: Necessary and sufficient conditions for the existence of Helmholtz decompositions in general domains. Annali dell' Universita di Ferrara 60 No 1 (2014) 245-262