Existence and uniqueness of invariant measure of dynamical system describing viscous Newtonian fluid

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Abstract.

Consider flow of homogeneous viscous incompressible Newtonian fluid in a compact and connected 3D volume V with fixed and smooth boundary ∂V. The fluid velocity will be represented by a vector field $\boldsymbol{v}(\boldsymbol{x})$ defined on V with no-slip boundary condition $\boldsymbol{v}(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \partial V$.

Let $\mathcal{L}_2(V)$ be the Hilbert space of real vector functions on V with $(\boldsymbol{u}, \boldsymbol{u}') = \int_V \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{u}'(\boldsymbol{x})\, dV/V$. Define $\mathcal{F}$ to be the closure in $\mathcal{L}_2(V)$ of the set of vector functions satisfying the zero-divergence and no-slip boundary conditions. Evidently $\boldsymbol{v} \in \mathcal{F}$.

Define basis $\{\boldsymbol{w}_I\}$ on $\mathcal{F}$ consisting of eigenfunctions of $\widetilde{\Delta}$, the projection of the Laplacian Δ from $\mathcal{L}_2(V)$ to $\mathcal{F}$. Then the velocity field $\boldsymbol{v}(\boldsymbol{x}) = v_I\, \boldsymbol{w}_I(\boldsymbol{x})$ (summation convention used) is represented by $(v_1, v_2, \dots) \in \ell_2$, which establishes isomorphism between Hilbert spaces $\mathcal{F}$ and ℓ_2. The characteristic wavelength of $\boldsymbol{w}_I(\boldsymbol{x})$ is $\omega_{(I)}^{-1}$ where $-\omega_{(I)}^2$ is the corresponding eigenvalue.

The Navier-Stokes equation (NSE) is equivalent to a system of ODE on ℓ_2

$$\frac{dv_I}{dt} = a_{I,JK}\, v_J\, v_K - b_{IJ}\, v_J + c_I \stackrel{\text{def}}{=} f_I(v) \qquad \text{where} \tag{1}$$

$$a_{I,JK} = -\int_V \boldsymbol{w}_I \cdot (\boldsymbol{w}_J \cdot \nabla)\boldsymbol{w}_K \,\frac{dV}{V}$$

$$b_{IJ} = -\vartheta \int_V \boldsymbol{w}_I \cdot \Delta \boldsymbol{w}_J \,\frac{dV}{V} = -\vartheta \int_V \boldsymbol{w}_I \cdot \widetilde{\Delta} \boldsymbol{w}_J \,\frac{dV}{V} = \vartheta \omega_{(I)}^2\, \delta_{IJ}$$

$$c_I = \int_V \boldsymbol{w}_I \cdot \boldsymbol{c}\,\frac{dV}{V} \qquad (\vartheta \text{ is fluid viscosity and } \boldsymbol{c} \text{ volume force})$$

The pressure term, zero-divergence equation and boundary conditions are all 'hidden' in the constant coefficients $a_{I,JK}$, b_{IJ} and c_I, each of which can be easily calculated numerically.

The equivalence between (1) and NSE holds only if all the basis vectors of $\mathcal{F}$ are used; then the number of degrees of freedom is infinite. However, for real-world fluids composed of molecules the macroscopic velocity $\boldsymbol{v}(\boldsymbol{x})$ cannot oscillate with a wavelength shorter than $\lambda_{\min}$ comparable with intermolecular distance. Therefore only a finite number of basis vectors $\boldsymbol{w}_I$ (those for which $\omega_{(I)}^{-1} > \lambda_{\min}$) can be used in the expansion $\boldsymbol{v}(\boldsymbol{x}, t) = v_I(t)\, \boldsymbol{w}_I(\boldsymbol{x})$ and the finite-dimensional system (1) is *not* equivalent to NSE. In other words, NSE is a valid representation of physical reality on macroscopic length scales but ceases to be valid at length scales for which the particle structure of matter becomes important.

We present a simple proof that for the dynamical system (1) with a finite dimension any solution is global and bounded, hence its invariant measure exists.

Also, the molecules of the fluid are in constant thermal motion. We show that this fact is mathematically described by a white noise term added to (1), thus changing it to an N-dimensional Itô stochastic differential equation. We prove that for such a system the invariant measure is unique.

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