On the Helmholtz decomposition of BMO spaces of vector fields

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The Helmholtz decomposition of vector fields is a fundamental tool for analysis of vector fields especially to analyze the Navier-Stokes equations in a domain. It gives a unique decomposition of a (tangential) vector field defined in a domain of an Euclidean space (or a Riemannian manifold) into a sum of a gradient field and a solenoidal field with supplemental condition like a boundary condition. It is well-known that such decomposition gives an orthogonal decomposition of the space of L^2 vector fields in an arbitrary domain and known as the Weyl decomposition. It is also well-studied that in various domains including the half space, smooth bounded and exterior domain, it gives a topological direct sum decomposition of the space of L^p vector fields for $1 < p < \infty$. The extension to the case $p=\infty$ (or $p=1$) is impossible because otherwise it would imply the boundedness of the Riesz type operator in L^∞ (or L^1) which is absurd.

In this talk, we extend the Helmholtz decomposition in a space of vector fields with bounded mean oscillations (BMO) when the domain of the vector field is a smooth bounded domain in an Euclidean space. There are several possible definitions of a BMO space of vector fields. However, to have a topological direct sum decomposition, it turns out that components of normal and tangential to the boundary should be handled separately.

This decomposition problem is equivalent to solving the Poisson equation with the divergence of the original vector field v as a data with the Neumann data with the normal trace of v. The desired gradient field is the gradient of the solution of this Poisson equation. To solve this problem we construct a kind of volume potential so that the problem is reduced to the Neumann problem for the Laplace equation. Unfortunately, taking the usual Newton potential causes a problem to estimate the necessary norm so we construct another volume potential based on normal coordinate. We need a trace theorem to control L^∞ norm of the normal trace. This is of independent interest. Finally, we solve the Neumann problem with L^∞ data in a necessary space. The Helmholtz decomposition for BMO vector fields is previously known only in the whole Euclidean space or the half space so this seems to be the first result for a domain with a curved boundary. This is a joint work with my student Z.Gu (University of Tokyo).